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GENERAL GROUP-THEORETIC TRANSFORMATIONS FROM BOUNDARY VALUE TO INITIAL VALUE PROBLEMS

Prepared under Contract No. NAS 8-20065 by
Tsung-Yen Na and Arthur G. Hansen

UNIVERSITY OF MICHIGAN

For

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER
Huntsville, Alabama

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UNIVERSITY OF MICHIGAN
Dearborn Campus

For

Space Sciences Laboratory

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER

FOREWORD

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ABSTRACT

A systematic method using S. Lie's continuous contact transformation groups is developed in this report which enables one to search for all possible groups of transformations under which a given differential equation can be transformed from a boundary value to an initial value problem. Examples are worked out in detail as illustrations of the procedure.

1. INTRODUCTION

The present report treats one of the most important applications of the concept of continuous transformation groups: the numerical solution of boundary value problems as related to the class of transformations from boundary value to initial value problems. A completely new and general method will be developed based on S. Lie's continuous and contact transformation group. In reference 6 the same concept was applied to develop a method of searching for all possible groups of transformations in a similarity analysis of partial differential equations.

A boundary value problem is characterized by the property that its boundary conditions are given at more than one point. In the absence of closed form solutions, numerical solutions must therefore be obtained by a trail-and-error procedure in which an unspecified boundary condition is assumed arbitrarily. The accuracy of the assumption is then checked by the fulfilling of the boundary condition at the other point.* It is therefore clear that the class of transformations from a boundary value to an initial value problem is of greatest importance in that it eliminates the trial-and-error procedure and simplifies considerably the process of numerical integration of the equation.

The first research on this type of transformations was given by Töpfer in 1912[7] for the numerical solution of the Blasius steady, two-dimensional boundary layer equations with uniform mainstream velocity. After a similarity transformation is made of the governing partial differential equations, a third-order non-linear ordinary differential equation is obtained with the boundary conditions specified at two points, namely, two at zero and one at infinity. The equation is then transcormed by Töpfer's method and the problem becomes an intial value problem. There seems to be little work on this subject until 1962 when Klamkin[2] published an important paper which considerably extended the range of applicability of the method, including applications to simutaneous ordinary differential equations. Both Töpfer and Klamkin's research consider the case in which boundary conditions are given at zero and infinity. No general theory, however, was given. Mostly recently, the method was reconsidered from the point of view of the theory of transformation groups in References 3 and 4. As a result, a general method was, indeed, developed for given groups of transformations. The method treated by Töpfer and Klamkin was found to be the special case of a linear group of transformations. Introduction of a "spiral group" of transformations made it possible to extend the method to a wide class of ordinary differential equations. According to this method the boundary conditions can be specified at both finite and infinite points. Extension of the method to problems in which two boundary conditions need to be transformed was also made by using a multi-parameter group of transformations. As long as the group of transformations is initially given,

*One would hope to eliminate this procedure by transforming all conditions to apply at one point. This is the method outlined here.

the method is straightforward. However, the arbitrariness in the selection of a practical group of transformations considerably limits the scope of application of the method. There is, therefore, a need to develop a method of searching for all possible groups of transformations for a given ordinary differential equation without resorting to an initial selection.

In the present report, a systematic method using S. Lie's continuous contact transofrmation groups will be developed which enables one to search for all possible groups of transformations under which the present method can be applied. The method used follows closely the general group-theoretic method given in Chapter 4 of Reference 6. A summary of the method given in References 3 and 4 will be given in the next section to summarize the present state of the research and the general concept of the method. The general method will be developed next, followed by two examples to show the steps which have to be taken to get specific groups of transformation applicable to a given problem.

2. SIMPLE GROUP—THEORETIC METHOD

2.0 FOREWORD

This article gives a critical review and summary of the method of transformation developed in References 3 and 4 with certain modifications. In section 2.1, the general concept of the method is illustrated by two examples, namely, the Blasius' problem from boundary-layer theory and the heat conduction equation with power-law heat generation.[5] The boundary conditions in the first example are given over an infinite interval. In the second example the method is applied to a case where the boundary conditions are given over a finite interval. A discussion is presented on the type of equations for which a spiral group is needed. In section 2.2, extension of the method to problems in which two boundary conditions need to be transformed is discussed in detail. Extension to more general types of equations is given in Section 2.3. Finally, Section 2.4 concludes the discussion of the method with an evaluation of its merits and limitations.

2.1 ONE-PARAMETER METHOD

2.1.1 Linear Group of Transformations

Consider the Blasius' equation mentioned earlier where we want to solve the equation

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0 \tag{2.1}$$

subject to the boundary conditions

$$f(0) = \frac{df(0)}{d\eta} = 0 , \quad \frac{df(\infty)}{d\eta} = 1$$

A one-parameter linear group of transformation

$$\eta = A^{\alpha_1}\overline{\eta} , \qquad f = A^{\alpha_2}\overline{f} \tag{2.2}$$

is applied to this equation, where A is the parameter of transformation and α_1 and α_2 are two constants to be determined. Under this transformation, Eq. (2.1) becomes

$$A^{\alpha_2-3\alpha_1} \frac{d^3\bar{f}}{d\bar{\eta}^3} + A^{2\alpha_2-2\alpha_1} \frac{1}{2} \bar{f} \frac{d^2\bar{f}}{d\bar{\eta}^2} = 0 \tag{2.3}$$

It is seen that the transformed equation, Eq. (2.3), will be independent of the parameter A if the powers of A in both terms are equal, i.e., if

$$\alpha_2 - 3\alpha_1 = 2\alpha_2 - 2\alpha_1 \tag{2.4}$$

Equation (2.3) then becomes

$$\frac{d^3\bar{f}}{d\bar{\eta}^3} + \frac{1}{2} \bar{f} \frac{d^2\bar{f}}{d\bar{\eta}^2} = 0 \tag{2.5}$$

From Eq. (2.4), we have

$$\alpha_2 = -\alpha_1 \tag{2.6}$$

This gives one relation between α_1 and α_2. The other equation required for the determination of α_1 and α_2 is given by putting

$$\frac{d^2f(0)}{d\eta^2} = A \tag{2.7}$$

from which

$$A^{\alpha_2-2\alpha_1} \frac{d^2\bar{f}(0)}{d\bar{\eta}^2} = A \tag{2.8}$$

Therefore, the transformed boundary condition will be independent of A if

$$\alpha_2 - 2\alpha_1 = 1 \tag{2.9}$$

which leaves

$$\frac{d^2\bar{f}(0)}{d\bar{\eta}^2} = 1 \tag{2.10}$$

The two unknown constants, α_1 and α_2, can then be obtained from Eqs. (2.6) and (2.9) as

$$\alpha_1 = -\alpha_2 = -\frac{1}{3} \tag{2.11}$$

Finally, the parameter of transformation, A, can be obtained by using the original boundary condition at infinity which gives

$$A^{\alpha_2-\alpha_1} \frac{d\bar{f}(\infty)}{d\bar{\eta}} = 1 \tag{2.12}$$

Therefore, we get

$$A = \left\{\frac{d\bar{f}(\infty)}{d\bar{\eta}}\right\}^{-3/2} \tag{2.13}$$

This example shows clearly the general concept of this technique. In general, two unknown constants (e.g., α_1 and α_2 in this example) are to be determined if one dependent variable is involved. Two equations for their determination are therefore necessary. One of these equations is obtained by requiring that the transformed ordinary differential equation be independent of the parameter of transformation, A; the other condition is formed by setting the original required boundary condition at the initial point equal to the parameter A. Finally, the parameter of transformation, A, is determined from the boundary condition at the other point. The solution of the problem then consists of two steps. In the solution of the Blasius equation, for example, Eq. (2.5) is first solved with the boundary conditions

$$\bar{f}(0) = \frac{d\bar{f}(0)}{d\bar{\eta}} = 0 , \quad \frac{d^2\bar{f}(0)}{d\bar{\eta}^2} = 1 \tag{2.14}$$

and the value of $d\bar{f}(\infty)/d\bar{\eta}$ is then obtained from the solution which in turn gives A by Eq. (2.13). With α_1, α_2 and A known, solution to Eq. (2.1) can be computed using Eq. (2.2). It is seen that the problem is reduced to an initial value problem.

With Blasius' equation treated in this way, extension to a new class of problems involving finite intervals becomes more obvious. We now consider the equation

$$\frac{d^2T}{dx^2} + \beta\, T^n = 0 \tag{2.15}$$

subject to the boundary conditions

$$T(0) = 0 , \quad T(L) = 0$$

This equation can be interpreted physically as heat conduction with power-law heat generation[5]

A one-parameter linear group of transformation

$$x = A^{\alpha_1}\bar{x} , \quad T = A^{\alpha_2}\bar{T} \tag{2.16}$$

is made and the two equations needed for the determination of α_1 and α_2 are obtained by requiring that: (1) the transformed equation to be independent of A; (2) by setting

$$\frac{dT(0)}{dx} = A \tag{2.17}$$

From these two equations, α_1 and α_2 are found to be

$$\alpha_1 = \frac{1-n}{2} , \quad \alpha_2 = 1 \tag{2.18}$$

The boundary condition, (2.17), becomes

$$\frac{d\bar{T}(0)}{d\bar{x}} = 1 \tag{2.19}$$

The parameter A is then found by transforming the boundary condition at x = L, which gives

$$\bar{T} = 0 \quad \text{at} \quad A^{\alpha_1}\bar{x} = L \tag{2.20a}$$

or,

$$A = \left\{\frac{L}{\bar{x}(\text{where } \bar{T} = 0)}\right\}^{2/(1-n)} \tag{2.20b}$$

To recapitulate, the procedure is therefore as follows: Firstly, the transformed equation is solved with the boundary conditions $\overline{T}(0) = 0$ and the condition given in (2.19). Then, the value of $\overline{x}$ where $\overline{T} = 0$ can be determined from the solution. The parameter of transformation, A, is finally computed from Eq. (2.20b). Again, the problem is reduced to an initial value problem.

Consider next the rather general second-order differential equation

$$\sum_{i=1}^{N} A_i \left(\frac{d^2y}{dx^2}\right)^{m_i} \left(\frac{dy}{dx}\right)^{n_i} y^{r_i} x^{s_i} = 0 \tag{2.21}$$

subject to the two cases of boundary conditions

Case I

$$y(0) = 0, \qquad \frac{d^d y(\infty)}{dx^d} = k$$

Case II

$$y(0) = 0, \qquad \frac{d^d y(L)}{dx^d} = 0$$

Consider the linear group of transformations

$$x = B^{\beta_1} \overline{x}, \qquad y = B^{\beta_2} \overline{y} \tag{2.22}$$

Under this group of transformation, Eq. (2.21) becomes

$$\sum_{i=1}^{N} B^{m_i(\beta_2 - 2\beta_1) + n_i(\beta_2 - \beta_1) + r_i\beta_2 + S_i\beta_1}$$

$$\times A_i \left(\frac{d^2\overline{y}}{d\overline{x}^2}\right)^{m_i} \left(\frac{d\overline{y}}{d\overline{x}}\right)^{n_i} \overline{y}^{r_i} \overline{x}^{s_i} = 0 \tag{2.23}$$

Equation (2.23) will be independent of the parameter of transformation, B, if the powers of B in each term are equal, i.e.,

$$m_1(\beta_2 - 2\beta_1) + n_1(\beta_2 - \beta_1) + r_1 \beta_2 + s_1 \beta_1$$

$$= m_i(\beta_2 - 2\beta_1) + n_i(\beta_2 - \beta_1) + r_i \beta_2 + S_i\beta_1 \tag{2.24}$$

where i = 2,...,N. In general, (2.24) gives (N-1) equations with only two unknowns, β_1 and β_2. The method is applicable only if the (N-1) equations actually reduce to one independent equation. To illustrate the problem that may arise, the Falkner-Skan equation may be cited.

$$\frac{d^3f}{d\eta^3} + \frac{1}{2}f\frac{d^2f}{d\eta^2} + \beta\left[1 - \left(\frac{df}{d\eta}\right)^2\right] = 0 \qquad (2.25)$$

Under the linear group of transformation defined by Eq. (2.22), Eq. (2.25) becomes

$$B^{\beta_2-3\beta_1}\frac{d^3\bar{f}}{d\bar{\eta}^3} + B^{2\beta_2-2\beta_1}\frac{1}{2}\bar{f}\frac{d^2\bar{f}}{d\bar{\eta}^2} + \beta\left[1 - B^{2\beta_2-2\beta_1}\left(\frac{d\bar{f}}{d\bar{\eta}}\right)^2\right] = 0 \qquad (2.26)$$

which is independent of the parameter of transformation if

$$\beta_2 - 3\beta_1 = 2\beta_2 - 2\beta_1 = 0 = 2\beta_2 - 2\beta_1 \qquad (2.27)$$

Two independent equations are obtained from (2.27). As a result, we get $\beta_1 = \beta_2 = 0$ which means the method is inapplicable.

Assuming for now that such a situation does not exist, Eq. (2.23) becomes

$$\sum_{i=1}^{N} A_i \left(\frac{d^2\bar{y}}{d\bar{x}^2}\right)^{m_i}\left(\frac{d\bar{y}}{d\bar{x}}\right)^{n_i}\bar{y}^{r_i}\,\bar{x}^{s_i} = 0 \qquad (2.28)$$

With one relation between β_1 and β_2 obtained from Eq. (2.24), the other relation required for the solution of β_1 and β_2 can be obtained by putting the slope at x = 0 equal to the parameter of transformation, B, i.e.,

$$\frac{dy(0)}{dx} = B \qquad (2.29)$$

After transformation, we have:

$$B^{\beta_2-\beta_1}\frac{d\bar{y}(0)}{d\bar{x}} = B$$

which is independent of B if

$$\beta_2 - \beta_1 = 1 \qquad (2.30)$$

The transformed boundary conditions are therefore,

$$\bar{y}(0) = 0 , \qquad \frac{d\bar{y}(0)}{d\bar{x}} = 1 \tag{2.31}$$

Equations (2.24) and (2.30) give solutions to β_1 and β_2.

Finally, the value of B can be found by applying the boundary condition at the second point. Thus:

Case I:

$$B^{\beta_2 - d\beta_1} \frac{d^d\bar{y}(\infty)}{d\bar{x}^d} = k$$

or,

$$B = \left\{ \frac{k}{\dfrac{d^d\bar{y}(\infty)}{d\bar{x}^d}} \right\}^{1/(\beta_2 - d\beta_1)} \tag{2.32}$$

Case II:

$$\frac{d^d\bar{y}}{d\bar{x}^d} = 0 \quad \text{at} \quad B^{\beta_1}\bar{x} = L$$

or,

$$B = \left\{ \frac{L}{x(\text{where } \dfrac{d^d\bar{y}}{d\bar{x}^d} = 0)} \right\}^{1/\beta_1} \tag{2.33}$$

Thus, Eq. (2.28) is solved with the boundary conditions given in Eq. (2.31) and the value of $\bar{x}$ where $d^d\bar{y}/d\bar{x}^d = 0$ can be found from the solution of the transformed equation. This result is then substituted into Eq. (2.33) and the value of B computed. It should be noted that there are cases where additional problems may arise. As an example, if the value of β in Eq. (2.15) is negative and also n = 1, then

$$\frac{d^2\bar{T}}{d\bar{x}^2} - \lambda^2\bar{T} = 0$$

with the boundary conditions

$$\bar{T}(0) = 0 , \qquad \frac{d\bar{T}(0)}{d\bar{x}} = 1$$

The solution is

$$\bar{T} = \frac{1}{\lambda} \sinh \lambda \bar{x}$$

which is never zero. This places another limitation on the method.

The boundary conditions at x = L in case II need not be homogeneous. For example, one may have

$$\text{at } x = L , \qquad \frac{d^d y}{dx^d} = k$$

Thus,

$$B^{\beta_2 - d\beta_1} \frac{d^d \bar{y}}{d\bar{x}^d} = k \quad \text{at} \quad B^{\beta_1} \bar{x} = L \tag{2.34}$$

Since k, L, β_1 and β_2 are known constants, the value of B can be found by searching for values of $\bar{x}$ and $d^d\bar{y}/d\bar{x}^d$ in the solution of Eq. (2.28) which give the same value of B in both equations of Eq. (2.34). One way of doing this is by eliminating B in Eq. (2.34) which leads to

$$\frac{d^d \bar{y}}{d\bar{x}^d} = k\left(\frac{L}{\bar{x}}\right)^{d - \frac{\beta_2}{\beta_1}} \tag{2.35}$$

Next, $(d^d\bar{y}/d\bar{x}^d)$ vs. $\bar{x}$ is plotted as a curve. Another curve from the solution to Eq. (2.28) can be plotted with the same coordinates. The intersection of these two curves will give the required value of $\bar{x}$ and $d^d\bar{y}/d\bar{x}^d$ which in turn can be used to compute B from Eq. (2.34).

One final remark about the method is necessary. Suppose the boundary condition at the initial point is $dy(0)/dx = 0$. In this case, we merely have to put

$$y(0) = B$$

Thus,

$$B^{\beta_2} \bar{y}(0) = B$$

and if the result is to be independent of B, β_2 must be equal to 1. Under no circumstances, however, should the boundary condition at the initial point be nonhomogeneous. If it is, one more equation relating β_1 and β_2 will result. The method then cannot be applied.

2.1.2 Spiral Group of Transformations

We now consider a class of nonlinear ordinary differential equations in which a spiral group of transformation rather than a linear group is needed for the method to apply. We consider here the class of equations

$$\sum_{i=1}^{N} C_i \left(\frac{d^2y}{dx^2}\right)^{m_i} \left(\frac{dy}{dx}\right)^{n_i} e^{p_i y}\, x^{q_i} = 0 \tag{2.36}$$

with the boundary conditions

Case I:

$$x = 0 : \quad \frac{dy}{dx} = 0 ; \qquad x = 1 : \quad y = 0$$

Case II:

$$x = 0 : \quad \frac{dy}{dx} = 0 ; \qquad x = \infty : \quad y = k_1$$

where C_i, m_i, n_i, p_i and q_i are constants and N is the number of terms in Eq. (2.36).

Let us define the one-parameter spiral group of transformations

$$x = e^{\beta_1 A}\bar{x} , \qquad y = \bar{y} + \alpha_2 A \tag{2.37}$$

where A is the parameter of transformation and α_1 and α_2 are constants to be determined.

Under this group of transformation, Eq. (2.36) becomes

$$\sum_{i=1}^{N} C_i e^{(-2m_i\alpha_1 - n_i\alpha_1 + p_i\alpha_2 + q_i\alpha_1)A}$$

$$\times \left(\frac{d^2\bar{y}}{d\bar{x}^2}\right)^{m_i} \left(\frac{d\bar{y}}{d\bar{x}}\right)^{n_i} e^{p_i\bar{y}}\, \bar{x}^{q_i} = 0 \tag{2.38}$$

The equation is seen to be independent of the parameter of transformation, A, if the powers of e in each term are equal, i.e.,

$$(-2m_i - n_i + q_i)\alpha_1 + p_i\,\alpha_2$$
$$= (-2m_1 - n_1 + q_1)\alpha_1 + p_1\,\alpha_2 \tag{2.39}$$

where i = 2, ..., N. The transformed equation becomes

$$\sum_{i=1}^{N} C_i \left(\frac{d^2\bar{y}}{d\bar{x}^2}\right)^{m_i} \left(\frac{d\bar{y}}{d\bar{x}}\right)^{n_i} e^{p_i\bar{y}}\,\bar{x}^{q_i} = 0 \tag{2.40}$$

Equation (2.39) represents (N-1) equations. In general, the method can be applied only if one independent equation results from these (N-1) equations. For example, if Eq. (2.36) takes the form

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + e^y = 0 \tag{2.41}$$

Equation (2.39) then gives one independent equation for α_1 and α_2 as

$$-2\alpha_1 = \alpha_2 \tag{2.42}$$

Physically, Eq. (2.41) may be interpreted as the equation for heat conduction in spheres with exponential heat generation[5].

To determine the second relation for the solution of α_1 and α_2, we put

$$y(0) = A \tag{2.43}$$

Upon transformation, this condition becomes:

$$\bar{y}(0) + \alpha_2 A = A$$

which is seen to be independent of A if

$$\alpha_2 = 1 \tag{2.44}$$

The transformed boundary conditions become

$$\bar{y}(0) = 0 \quad \text{and} \quad \frac{d\bar{y}(0)}{d\bar{x}} = 0 \tag{2.45}$$

For the example given in Eq. (2.41), α_1 and α_2 can be found from Eqs. (2.42) and (2.44):

$$\alpha_1 = -\frac{1}{2} \quad \text{and} \quad \alpha_2 = 1$$

Finally, to get the parameter of transformation, A, the boundary condition at the second point is used. The two cases are considered separately.

Case I. The boundary condition at $x = 1$ becomes

$$\bar{y} + A = 0 \quad \text{at} \quad e^{\alpha_1 A}\bar{x} = 1 \tag{2.46}$$

Eliminating A, we get

$$e^{-\alpha_1 \bar{y}}\,\bar{x} = 1 \tag{2.47}$$

Case II. The boundary condition at $x = \infty$ becomes

$$\bar{x} = \infty: \quad \bar{y} + \alpha_2 A = k_1$$

or

$$A = \frac{1}{\alpha_2}\left[k_1 - \bar{y}(\infty)\right] \tag{2.48}$$

Therefore, the method proceeds as follows: First, Eq. (2.40) is solved with the boundary conditions (2.45). In case I, the solution curve to Eq. (2.40) can be plotted on $\bar{y}$ vs. $\bar{x}$ coordinates. Equation (2.47) is plotted on the same coordinates. The intersection of these two curves gives the values of $\bar{y}$ and $\bar{x}$ which give the same value of A from Eq. (2.46). The value of A is then determined. In case II, the value of A can be computed from Eq. (2.48).

The solution to the original equation, Eq. (2.36) can be obtained from Eq. (2.37) since now α_1, α_2, and A are known constants.

2.2 TWO PARAMETER METHOD

2.2.1 Transformation of Two Boundary Conditions

The method developed in the preceding section can be extended to higher-order differential equations as long as only one boundary condition is required to be transformed. In this section and the next section, the method will be extended to higher-order differential equations in which more than one boundary condition need to be transformed. For such cases, multiparameter groups of transformations are required.

Consider now the third order differential equation

$$\sum_{i=1}^{N} A_i \left(\frac{d^3y}{dx^3}\right)^{m_i} \left(\frac{d^2y}{dx^2}\right)^{n_i} \left(\frac{dy}{dx}\right)^{r_i} y^{s_i} x^{t_i} = 0 \tag{2.49}$$

subject to the boundary conditions

Case I.

$$y(0) = 0 \ , \quad \frac{d^{d_1}y(\infty)}{dx^{d_1}} = k_1 \ , \quad \frac{d^{d_2}y(\infty)}{dx^{d_2}} = k_2$$

Case II.

$$y(0) = 0, \quad \frac{d^{d_1}y(L)}{dx^{d_1}} = 0 \ , \quad \frac{d^{d_2}y(L)}{dx^{d_2}} = k_2$$

We now define a two-parameter group of transformation

$$x = B^{\beta_1} C^{\gamma_1} \bar{x} \ , \qquad y = B^{\beta_2} C^{\gamma_2} \bar{y} \tag{2.50}$$

Under this group of transformation, Eq. (2.49) becomes

$$\sum_{i=1}^{N} A_i \ B^{m_i(\beta-3\beta_1)+n_i(\beta_2-2\beta_1)+r_i(\beta_2-\beta_1)+S_i\beta_2+t_i\beta_1}$$

$$\times \ C^{m_i(\gamma_2-3\gamma_1)+n_i(\gamma_2-2\gamma_1)+r_i(\gamma_2-\gamma_1)+S_i\gamma_2+t_i\gamma_1}$$

$$\times \left(\frac{d^3\bar{y}}{d\bar{x}^3}\right)^{m_i} \left(\frac{d^2\bar{y}}{d\bar{x}^2}\right)^{n_i} \left(\frac{d\bar{y}}{d\bar{x}}\right)^{r_i} \bar{y}^{S_i} \bar{x}^{t_i} = 0 \tag{2.51}$$

The method can be applied if, for all i's,

$$m_i(\beta_2-3\beta_1)+n_i(\beta_2-2\beta_1)+r_i(\beta_2-\beta_1)+S_i\beta_2+t_i\beta_1 = C_1 \qquad (2.52a)$$

$$m_i(\gamma_2-3\gamma_1)+n_i(\gamma_2-2\gamma_1)+r_i(\gamma_2-\gamma_1)+S_i\gamma_2+t_i\gamma_1 = C_2 \qquad (2.52b)$$

where C_1 and C_2 are two arbitrary constants.

Equation (2.51) then becomes

$$\sum_{i=1}^{N} A_i \left(\frac{d^3\bar{y}}{d\bar{x}^3}\right)^{m_i} \left(\frac{d^2\bar{y}}{d\bar{x}^2}\right)^{n_i} \left(\frac{d\bar{y}}{d\bar{x}}\right)^{r_i} \bar{y}^{S_i}\, \bar{x}^{t_i} = 0 \qquad (2.53)$$

For example, the equation

$$\frac{d^3y}{dx^3} + \sqrt{\frac{1}{x^3}\,\frac{dy}{dx}\,\frac{d^2y}{dx^2} + \left(\frac{y}{x^3}\right)^2} = 0 \qquad (2.54)$$

belongs to this class.

The boundary condition at the initial point, $y(0) = 0$, can be transformed to

$$\bar{y}(0) = 0 \qquad (2.55)$$

To get the other boundary conditions at the initial point, let us put

$$\frac{dy(0)}{dx} = B \quad \text{and} \quad \frac{d^2y(0)}{dx^2} = C \qquad (2.56)$$

Upon transformation, (2.56) becomes

$$B^{\beta_2-\beta_1}\, C^{\gamma_2-\gamma_1}\, \frac{d\bar{y}(0)}{d\bar{x}} = B \qquad (2.57a)$$

and

$$B^{\beta_2-2\beta_1}\, C^{\gamma_2-2\gamma_1}\, \frac{d^2\bar{y}(0)}{d\bar{x}^2} = C \qquad (2.57b)$$

which are seen to be independent of B and C if, from (2.57a),

$$\beta_2 - \beta_1 = 1 , \qquad \gamma_2 - \gamma_1 = 0$$

and, from (2.57b),

$$\beta_2 - 2\beta_1 = 0 , \qquad \gamma_2 - 2\gamma_1 = 1 .$$

(2.57a) and (2.57b) are then transformed to

$$\frac{d\bar{y}(0)}{d\bar{x}} = 1 , \qquad \frac{d^2\bar{y}(0)}{d\bar{x}^2} = 1 \tag{2.58}$$

and the values of β_1, β_2, γ_1 and γ_2 are

$$\beta_1 = 1, \ \beta_2 = 2, \ \gamma_1 = -1, \ \gamma_2 = -1$$

To get the parameters of transformation, B and C, the boundary conditions at the second point are used.

Case I

$$B^{2-d_1} \, C^{d_1-1} = \frac{k_1}{\dfrac{d^{d_1}\bar{y}(\infty)}{d\bar{x}^{d_1}}} \tag{2.59a}$$

$$B^{2-d_2} \, C^{d_2-1} = \frac{k_2}{\dfrac{d^{d_2}\bar{y}(\infty)}{d\bar{x}^{d_2}}} \tag{2.59b}$$

Case II

$$BC^{-1} = \frac{L}{\bar{x} \ \left(\text{where } \dfrac{d^{d_1}\bar{y}}{d\bar{x}^{d_1}} = 0\right)} \tag{2.60a}$$

$$B^{2-d_2} \, C^{-1+d_2} = \frac{k_2}{\dfrac{d^{d_2}\bar{y}}{d\bar{x}^{d_2}}} \tag{2.60b}$$

Therefore, B and C can be solved from Eqs. (2.59) or (2.60).

The method can easily be extended to equations of the type:

$$\sum_{i=1}^{N} A_i \left(\frac{d^3y}{dx^3}\right)^{m_i} \left(\frac{d^2y}{dx^2}\right)^{n_i} \left(\frac{dy}{dx}\right)^{r_i} e^{s_i y}\, x^{t_i} = 0 \qquad (2.61)$$

The only difference here is that one assumes a transformation group defined by

$$x = e^{\beta_1 B + \gamma_1 C}\,\bar{x}\,, \qquad y = \bar{y} + \beta_2 B + \gamma_2 C \qquad (2.62)$$

Other steps remain the same.

2.2.2 Simutaneous Differential Equations

Application of the method to simutaneous differential equations again involves mutli-parameter groups. Consider now the following system of two simutaneous equations:

$$\sum_{i=1}^{N} A_i \left(\frac{d^2y}{dx^2}\right)^{m_i} \left(\frac{dy}{dx}\right)^{n_i} y^{p_i} \left(\frac{d^2z}{dx^2}\right)^{r_i} \left(\frac{dz}{dx}\right)^{S_i} z^{t_i}\, x^{q_i} = 0 \qquad (2.63a)$$

$$\sum_{j=1}^{M} B_j \left(\frac{d^2y}{dx^2}\right)^{\bar{m}_j} \left(\frac{dy}{dx}\right)^{\bar{n}_j} y^{\bar{p}_j} \left(\frac{d^2z}{dx^2}\right)^{\bar{r}_j} \left(\frac{dz}{dx}\right)^{\bar{S}_j} z^{\bar{t}_j}\, x^{\bar{q}_j} = 0 \qquad (2.63b)$$

subject to the boundary conditions

Case I.

$$y(0) = 0,\ z(0) = 0,\ \frac{d^d y(\infty)}{dx^d} = k_1\,, \quad \frac{d^{\bar{d}} z(\infty)}{dx^{\bar{d}}} = k_2$$

Case II.

$$y(0) = 0,\ z(0) = 0,\ \frac{d^d y(L)}{dx^d} = 0\,, \quad \frac{d^{\bar{d}} z(L)}{dx^{\bar{d}}} = k_2$$

Let us now define a two-parameter transformation group

$$x = {}^{\beta_1}\,\bar{x}\,, \quad y = \lambda^{\beta_2}\bar{y}\,, \quad z = \lambda^{\beta_3}\mu^{\delta}\bar{z} \qquad (2.64)$$

Again, the differential equations, (2.63), are independent of the parameters of transformation, λ and μ, if the powers of λ and μ in each term are respectively the same. This leads to the following system of equations for the solution of β_1, β_2, β_3 and δ:

$$(\beta_2-2\beta_1)m_i + (\beta_2-\beta_1)n_i + \beta_2 p_i + r_i(\beta_3-2\beta_1) + S_i(\beta_3-\beta_1) + t_i \beta_3 + q_i\beta_1$$

$$= (\beta_2-2\beta_1)m_1 + (\beta_2-\beta_1)n_1 + \beta_2 p_1 + r_1(\beta_3-2\beta_1) + S_1(\beta_3-\beta_1) + t_1 \beta_3 + q_1 \beta_1 \tag{2.65}$$

$$r_i + S_i + t_i = r_1 + S_1 + t_1 \tag{2.66}$$

$$(\beta_2-2\beta_1)\bar{m}_j + (\beta_2-\beta_1)\bar{n}_j + \beta_2 \bar{p}_j + \bar{r}_j(\beta_3-2\beta_1) + \bar{S}_j(\beta_3-\beta_1) + \bar{t}_j \beta_3 + \bar{q}_j \beta_1$$

$$= (\beta_2-2\beta_1)\bar{m}_1 + (\beta_2-\beta_1)\bar{n}_1 + \beta_2 \bar{p}_1 + \bar{r}_1 (\beta_3-2\beta_1) + \bar{S}_1(\beta_2-\beta_1) + \bar{t}_1\beta_3 + \bar{q}_1 \beta_1 \tag{2.67}$$

$$\bar{r}_j + \bar{S}_j + \bar{t}_j = \bar{r}_1 + \bar{S}_1 + \bar{t}_1 \tag{2.68}$$

where $i = 2,\ldots, N$ and $j = 2,\ldots,M$.

Substitution of Eqs. (2.66) and (2.68) into Eqs. (2.65) and (2.67), resepctively, gives

$$(m_i+n_i+p_i)\beta_2 - (2m_i+n_i+2r_i+S_i-q_i)\beta_1$$
$$= (m_1+n_1+p_1)\beta_2 - (2m_1+n_1+2r_1+S_1-q_1)\beta_1 \tag{2.69}$$

$$(\bar{m}_j+\bar{n}_j+\bar{p}_j)\beta_2 - (2\bar{m}_j+\bar{n}_j+2\bar{r}_j+\bar{S}_j-\bar{q}_j)\beta_1$$
$$= (\bar{m}_1+\bar{n}_1+\bar{p}_1)\beta_2 - (2\bar{m}_1+\bar{n}_1+ 2\bar{r}_1 + \bar{S}_1 - \bar{q}_1)\beta_1 \tag{2.70}$$

The method is applicable if Eqs. (2.69) and (2.70) each represent only one independent equation and that both give the same ratio of β_2/β_1. If these conditions are satisfied, the ratio of β_2/β_1 is known.

Next, the required boundary contions are defined to be equal to λ and μ respectively, i.e.,

$$y'(0) = \lambda \quad \text{and} \quad z'(0) = \mu$$

Upon transformation,

$$\lambda^{\beta_2-\beta_1}\bar{y}'(0) = \lambda \quad \text{and} \quad \mu^{\delta}\,\lambda^{\beta_3-\beta_1}\bar{z}'(0) = \mu$$

which then give

$$\beta_2 - \beta_1 = 1 \,, \quad \delta = 1 \quad \text{and} \quad \beta_3 - \beta_1 = 0 \tag{2.71}$$

The ratio of β_2/β_1 obtained from Eqs. (2.69) and (2.70), together with Eq. (2.71), gives solutions of β_1, β_2, β_3 and δ.

To get the parameters of transformation, the same method discussed in previous paragraphs can be applied. It will not be repeated here.

The method can be easily generalized to include cases with exponentials of y or z or both in Eqs. (2.63).

2.3 MORE GENERAL TYPES OF EQUATIONS

The method developed above can be extended to more general types of equation. Two cases are considered here. Consider now the general second order differential equations:

$$G\left\{\sum_{j=1}^{M} b_j \left(\frac{d^2y}{dx^2}\right)^{t_j} \left(\frac{dy}{dx}\right)^{s_j} y^{r_j}\, x^{u_j}\right\} = 0 \tag{2.72}$$

where G represents an arbitrary function of the argument indicated.

Case I. The method can be applied if, under the linear transformation group

$$x = A^{\alpha_1}\,\bar{x} \,, \qquad y = A^{\alpha_2}\,\bar{y},$$

only one relation between α_1 and α_2 is obtained from the condition that Eq. (2.72) is invariant under this group of transformation.

As an example, the equation

$$y^3 \frac{d^2y}{dx^2} + \sin\left(y \frac{dy}{dx}\right) + 1 = 0$$

will give one relation between α_1 and α_2, namely,

$$2\alpha_2 - \alpha_1 = 0$$

Case II. If y is absent in all terms in Eq. (2.72), the spiral group of transformation can always be applied for any arbitrary function in Eq. (2.72). As an example, consider

$$\frac{d^2y}{dx^2} + f_1\left(\frac{dy}{dx}\right) + 1 = 0 \tag{2.73}$$

where f_1 is any arbitrary function of dy/dx. Under the spiral group of transformation

$$x = e^{\alpha_1 a}\,\bar{x}, \qquad y = \bar{y} + \alpha_2 a$$

Eq. (2.73) becomes

$$e^{-2\alpha_1 a} \frac{d^2\bar{y}}{d\bar{x}^2} + f_1\left(e^{-\alpha_1 a} \frac{d\bar{y}}{d\bar{x}}\right) + 1 = 0$$

which is independent of a if $\alpha_1 = 0$ for any arbitrary function f_1. The remaining steps remain the same.

2.4 CONCLUDING REMARKS

In this section, the application of a linear or spiral group of transformation to the class of transformation from a boundary value to an initial value problem is treated. The method consists of three basic steps. First, a transformation group is defined and the given differential equation is required to be invariant, i.e., independent of the parameter of transformation, under this group of transformation. In step 2, the required boundary condition is set to be equal to the parameter of transformation. Finally, the parameter of transformation is found by using the boundary condition at the second point. Knowing this general concept, the method treated in this article can be applied to higher-order equations or other types of equations. It is simple to apply and only algebraic solutions are required to get the transformation. The main disadvantage of this method lies, however, on the arbitrariness in the selction of a proper group for a given differential equation. In the next two sections, a very general method will be developed which makes it possible to search for all possible groups of transformation under which the given differential equation can be reduced to an initial value problem.

3. GENERAL GROUP-THEORETIC METHOD

3.0 GENERAL CONSIDERATION

In order to introduce the general group-theoretic method, the method developed in section 2 will be summarized by considering a second-order ordinary differential equation as follows:

Consider the ordinary differential equation

$$F\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y, x\right) = 0 \tag{3.1}$$

with the boundary conditions

$$y(0) = 0 , \qquad y(\ell) = a.$$

The differential equation is transformed by introducing a one-parameter group of transformation, viz.,

$$x = f(\overline{x}, A, \alpha_1, \alpha_2)$$

$$y = g(\overline{y}, A, \alpha_1, \alpha_2) \tag{3.2}$$

where α_1 and α_2 are constants to be determined before the transformed equation is solved, and A is the parameter of transformation to be determined after it is solved.

To determine α_1 and α_2, two conditions are imposed:

i. the given differential equation is to be invariant; i.e., it should be independent of the parameter of transformation, A; and
ii. the boundary condition $d\overline{y}(0)/d\overline{x}$ is to be independent of A for some choice of $dy(0)/dx$ as a function of A.

If α_1 and α_2 can be found satisfying the above conditions, the method can proceed

The transformed differential equation can now be solved as an initial value problem with the initial conditions $\overline{y}(0) = 0$ and $d\overline{y}(0)/d\overline{x} = b$, where b is the value resulting from condition ii. If the solution of the initial value problem is denoted by $\bar{y} = b(\bar{x})$, the value of A needed for the completion of the solution of the original equation is sought by solving the following system of equations:

$$\bar{y} = h(\bar{x}) \; , \quad \ell = f(\bar{x}, A, \alpha_1, \alpha_2) \; , \quad a = g(\bar{y}, A, \alpha_1, \alpha_2) \tag{3.3}$$

The last two equations come from the boundary condition at $x = \ell$. The method fails if no values of A can be found from Eq. (3.3).

The key steps in the above scheme are the selection of a specific group of transformations and the requirement that the given differential equation be invariant under this group of transformations. For a given differential equation, the equation may not be invariant under a specific preassigned group of transformations. This does not rule out that it will always so if other groups are introduced. It is therefore clear that a method of searching for possible groups under which the given differential equation be invariant is of great importance. To achieve this goal, the two steps mentioned above are reversed. One starts by requiring that the given differential equation be invariant under ar "infinitesimal transformation". The resulting equation is then used to search for the possible groups of transformation which satisfy this requirement. This necessitates a brief review of those concepts given in Reference 6 which are related to this method.

3.1 THE INFINITESIMAL CONTACT TRANSFORMATION*

3.1.1 Infinitesimal Transformation

Let the identical transformation be

$$\begin{aligned} \phi(x,y,a_o) &= x \\ \psi(x,y,a_o) &= y \end{aligned} \tag{3.4}$$

then the transformation

$$\begin{aligned} x_1 &= \phi(x,y,a_o+\delta\epsilon) \\ &= \phi(x,y,a_o) + \frac{\delta\epsilon}{1!}\left(\frac{\partial\phi}{\partial a}\right)_{a_o} + \frac{\partial\epsilon^2}{2!}\left(\frac{\partial^2\phi}{\partial a^2}\right)_{a_o} + \text{-----} \end{aligned} \tag{3.5}$$

*For detail, the reader is referred to Reference 6.

$$y = \psi(x,y,a_o+\delta\tau)$$

$$= \psi(x,y,a_o) + \frac{\delta\epsilon}{1!}\left(\frac{\partial\psi}{\partial a}\right)_{a_o} + \frac{\delta\epsilon^2}{2!}\left(\frac{\partial^2\psi}{\partial a^2}\right)_{a_o} + \text{------} \qquad (3.6)$$

assuming $\delta\epsilon$ is infinitesimal, neglecting higher order terms of $\delta\epsilon$ and using the relation for the identical transformation, we put

$$x_1 = x + \xi(x,y)\,\delta\epsilon$$

$$y_1 = y + \eta(x,y)\,\delta\epsilon \qquad (3.7)$$

3.1.2 Notation for the Infinitesimal Transformation

The employment of the infinitesimal transformation

$$x_1 = x + \xi\,\delta\epsilon \quad \text{and} \quad y_1 = y + \eta\,\delta\epsilon \qquad (3.8)$$

in conjunction with the function $f(x,y)$ will be to transform $f(x,y)$ into $f(x_1,y_1)$ which upon expanding in Taylor series, becomes

$$f(x_1,y_1) = f(x+\xi\delta\epsilon,\ y + \eta\delta\epsilon)$$

$$= f(x,\eta) + \frac{\delta\epsilon}{1!}\left(\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y}\right)$$

$$+ \frac{\delta\epsilon^2}{2!}\left(\xi^2\frac{\partial^2 f}{\partial x^2} + 2\xi\eta\frac{\partial^2 f}{2x2y} + \eta^2\frac{\partial^2 f}{\partial y^2}\right)$$

$$+ \text{---------------------------}$$

$$+ \frac{\partial\epsilon^n}{n!}\left(\xi^n\frac{\partial^n f}{\partial x^n} + c_1^n\,\xi^{n-1}\,\eta\,\frac{\partial^n f}{\partial x^{n-1}\partial y} + \text{-------}\right.$$

$$\left.\text{---------} + \eta^n\frac{\partial^n f}{\partial y^n}\right)$$

$$= f(x,y) + \frac{\partial\epsilon}{1!}Uf + \frac{\delta\epsilon^2}{2!}U^2f + \text{----------} \qquad (3.9)$$

where

$$Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} \tag{3.10}$$

is called the group representation and $U^n f$ means repeating the operator U for n times.

3.1.3. Invariant Function

If $f(x_1,y_1) = f(x,y)$, then f is invariant under the infintesimal transformation.

Theorem. The necesary and sufficient condition that $f(x,y)$ be invariant under the group represented by Uf is $Uf = 0$; i.e.,

$$\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} = 0 \tag{3.11}$$

To solve for the invariant function, we solve the related differential equation

$$\frac{dx}{\xi} = \frac{dy}{\eta} \tag{3.12}$$

If the solution is

$$\Omega(x,y) = \text{constant} \tag{3.13}$$

this function is the invariant function for the infintesimal transformation represented by Uf. Since Eq. (3.12) has only one independent solution depending on a simple arbitrary constant, a one-parameter group in two variables has one and only one independent invariant.

3.1.4. Extension to n variable

The condition for $f(x_1,\ldots,x_n)$ to be invariant under an infinitesimal transformation is

$$Uf = \xi_1(x_1,\ldots,x_n) \frac{\partial f}{\partial x_1} + \text{--------} \\ \text{----} + \xi_n(x_1,\ldots,x_n) \frac{\partial f}{\partial x_n} = 0 \tag{3.14}$$

To get invariant functions, we solve

$$\frac{dx_1}{\xi_1} = \text{--------------} = \frac{dx_n}{\xi_n} \qquad (3.15)$$

Since there exist (n-1) independent solutions, a one-parameter group in n variables has (n-1) indpendent invariants.

3.1.5 Invariance of an Ordinary Differential Equation

Consider a kth-order ordinary differential equation

$$F(x,\ y,\ y',\ y'',\ \ldots,y^{(k)}) = 0 \qquad (3.16)$$

This equation is invariant under the infitesimal transformation defined by

$$\begin{aligned}
\bar{x} &= x + \delta\epsilon\ \xi(x,\ y,\ y') \\
\bar{y} &= y + \delta\epsilon\ \eta(x,\ y,\ y') \\
\bar{y}' &= y' + \delta\epsilon\ \pi_1\ (x,\ y,\ y') \\
&\text{----------} \\
&\text{----------} \\
\bar{y}^{(k)} &= y^{(k)} + \delta\epsilon\ \pi_k\ (x,\ y,\ y',\ldots,y^{(k)}) \qquad (3.17)
\end{aligned}$$

if the following condition is satisfied:

$$UF = 0 \qquad (3.18a)$$

or, in expanded form,

$$\xi\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y} + \pi_1\frac{\partial F}{\partial y'} + \text{------} + \pi_k\frac{\partial F}{\partial y^{(k)}} = 0 \qquad (3.18b)$$

For a given group of transformation, the functions ξ, η, $\pi_1,\ldots,$ π_k are known. Equation (3.18) gives the condition which the given differential equation, Eq. (3.16), must satisfy if it can be transformed to an initial value problem. However, if the group of transformation is not given, Eq. (3.18) alone will not be enough to search for possible groups. At this point, the theories developed in Reference 6 on the concept of an infinitesimal contact transformation must be introduced which will ultimately makes it possible to express these functions in terms of the so-called "characteristic function".

3.1.6 Definition of a Contact Transformation*

When Z, X_1, $X_2, \ldots, X_n$, $P_1, \ldots, P_n$ are $2n+1$ independent functions of the $2n+1$ independent quantities z_1, $x_1, \ldots, x_n$, $p_1, \ldots, p_n$ such that the relation

$$dZ - P_i \, dX_i = \rho(dz - p_i dx_i) \tag{3.19}$$

(where p does not vanish) is identically satisfied, then the transformation defined by the equations

$$z' = Z, \; x' = X, \; p' = P \tag{3.20}$$

is called a contact transformation.

3.1.7. Infinitesimal Contact Transformation

From Eq. (3.19),

$$\begin{aligned} &\frac{\partial Z}{\partial z} dz + \frac{\partial Z}{\partial x_i} dx_i + \frac{\partial Z}{\partial p_i} dp_i \\ &\quad - P_i\left(\frac{\partial X_i}{\partial z} dz + \frac{\partial X_i}{\partial x_r} dx_r + \frac{\partial X_i}{\partial p_r} dp_r\right) \\ &\quad = \rho(dz - p_i \, dx_i) \end{aligned} \tag{3.21}$$

For the infinitesimal transformation

$$Z = z + \delta\epsilon\xi; \quad X_i = x_i + \delta\epsilon \, \xi_i, \quad P_i = p_i + \delta\epsilon \, \pi_i \tag{3.22}$$

we get

*S. Lie, Math. Ann., t. viii, p. 220

$$\frac{\partial\zeta}{\partial z} - p_i \frac{\partial\xi_i}{\partial z} = \sigma$$

$$\frac{\partial\zeta}{\partial p_r} - p_i \frac{\partial\xi_i}{\partial p_r} = 0$$

$$\frac{\partial\zeta}{\partial x_r} - p_i \frac{\partial\xi_i}{\partial x_r} - \pi_r = -\sigma p_r \qquad (3.23)$$

If a characteristic function, $\overline{W}$, is defined as $\overline{W} = p_i\xi_i - \zeta$, then

$$\xi_r = \frac{\partial\overline{W}}{\partial p_r}, \qquad \zeta = p_i \frac{\partial\overline{W}}{\partial p_i} - \overline{W},$$

$$\pi_r = -\frac{\partial\overline{W}}{\partial x_r} - p_r \frac{\partial\overline{W}}{\partial z} \qquad (3.24)$$

Higher order transformation functions, π_{ij}, π_{ijk}, etc., can also be expressed in terms of $\overline{W}$. However, due to the complexity in their derivation, they will not be included here. For detail the reader is refered to Reference 6. However, a special case with one independent and one dependent variable will be given here since it will be needed in the next section. For this case, we consider an infinitesimal transformation

$$x' = x + (\delta\epsilon)\,\xi(x, y, p)$$

$$y' = y + (\delta\epsilon)\,\Theta(x, y, p)$$

$$p' = p + (\delta\epsilon)\,\pi(x, y, p)$$

$$q' = q + (\delta\epsilon)\,k(x, y, p, q)$$

$$r' = r + (\delta\epsilon)\,\rho(x, y, p, q, r) \qquad (3.25)$$

where $p = dy/dx$, $q = d^2y/dx^2$ and $r = d^3y/dx^3$. The transformation functions can be expressed in terms of a characteristic function W as[6]:

$$\xi = \frac{\partial\overline{W}}{\partial p}$$

$$\Theta = p\frac{\partial\overline{W}}{\partial p} - \overline{W}$$

$$-\pi = X\overline{W}$$

$$-k = (X^2 + 2qX \frac{\partial}{\partial p} + q^2 \frac{\partial^2}{\partial p^2} + q \frac{\partial}{\partial f}) \overline{W}$$

$$-\rho = (X^3 + 3q\, X^2 \frac{\partial}{\partial p} + 3q^2 X \frac{\partial^2}{\partial p^2} + q^3 \frac{\partial^3}{\partial p^3} + 3qX \frac{\partial}{\partial f} + 3q^2 \frac{\partial^2}{\partial f \partial p}) \overline{W}$$

$$+ r(3q \frac{\partial^2}{\partial p^2} + 3X \frac{\partial}{\partial p} + \frac{\partial}{\partial f}) \overline{W} \qquad (3.26)$$

where the operator $X = \partial/\partial x + p\, \partial/\partial y$.

3.2 THE GENERAL METHOD

With the background discussed in Sections 3.0 and 3.1 in mind, the second-order ordinary differential equation is again used to illustrate the steps for the transformation from a boundary value to an initial value problem and the search of possible groups to achieve the transformation.

Consider again Eq. (3.1), the method proceeds as follows:

i. An infitesimal transformation is defined, as in Eq. (3.25) except the transformation for r is not needed here. The given differential equation, Eq. (3.1), is required to be invariant under this group of transformation, i.e., it must satisfy Eq. (3.18).

ii. The transformation functions can be expressed as a function of the characteristic function, $\overline{W}$, as given in Eq. (3.26). Eq. (3.18) now becomes an equation with an unknown function $\overline{W}$. The functional form of $\overline{W}$ can be predicted.

iii. After $\overline{W}$ is known, the transformation functions become known functions and the finite form of the transformation can be derived by Eq. (3.9).

Two examples will be given in the next section.

4. APPLICATION OF THE GENERAL GROUP—THEORETIC METHOD

In this section, the general theories given in section 3 will be applied to two examples, namely, the Falkner-Skan problem and the heat conduction equation with non-linear heat generation. These examples serve only as illustrations of the method.

4.1 APPLICATION TO FALKNER-SKAN SOLUTIONS

Consider the well-known Falkner-Skan differential equation from boundary layer theory[1]:

$$f''' + f\,f'' + \beta(1 - f'^2) = 0 \tag{4.1}$$

The boundary conditions are

$$f(0) = f'(0) = 0\,; \quad f'(\infty) = 1$$

We now use the notations

$$p = f'\,, \quad q = f''\,, \quad r = f''' \tag{4.2}$$

then Eq. (4.1) and its boundary conditions become

$$r + f\,q + \beta\,(1 - p^2) = 0 \tag{4.3}$$

with boundary conditions

$$f(0) = p(0) = 0\,; \quad p(\infty) = 1$$

Next, an infinitesimal transformation is defined as[6]

$$\eta' = \eta + (\delta\epsilon)\,\xi(\eta, f, p)$$

$$f' = f + (\delta\epsilon)\,\Theta(\eta, f, p)$$

$$p' = p + (\delta\epsilon)\,\pi(\eta,\ f,\ p)$$

$$q' = q + (\delta\epsilon)\,k(\eta,\ f,\ p,\ q)$$

$$r' = r + (\delta\epsilon)\,\rho(\eta,\ f,\ p,\ q,\ r) \tag{4.4}$$

where, in terms of the characteristic function, W,

$$\xi = \frac{\partial\overline{W}}{\partial p}$$

$$\theta = p\,\frac{\partial\overline{W}}{\partial p} - \overline{W}$$

$$-\pi = X\,\overline{W}$$

$$-k = \left(X^2 + 2q\,X\,\frac{\partial}{\partial p} + q^2\,\frac{\partial^2}{\partial p^2} + q\,\frac{\partial}{\partial f}\right)\overline{W}$$

$$-\rho = \left(X^3 + 3q\,X^2\,\frac{\partial}{\partial p} + 3q^2\,X\,\frac{\partial^2}{\partial p^2} + q^3\,\frac{\partial^3}{\partial p^3} + 3q\,X\,\frac{\partial}{\partial f} + 3q^2\,\frac{\partial^2}{\partial f\partial p}\right)\overline{W}$$

$$+\ r\left(3q\,\frac{\partial^2}{\partial p^2} + 3X\,\frac{\partial}{\partial p} + \frac{\partial}{\partial f}\right)\overline{W} \tag{4.5}$$

The operator X in Eq. (4.5) is defined as

$$X = \frac{\partial}{\partial\eta} + p\,\frac{\partial}{\partial f} \tag{4.6}$$

According to the theory discussion in the previous article, the condition imposed on the differential equation is that it is independent of the parameter of transformation, ϵ, under the transformation defined by Eq. (4.4), i.e.,

$$\frac{\partial F}{\partial\epsilon} = 0 \tag{4.7}$$

where F represents the differential equation, (4.3). Equation (4.7) can be written in its expanded form as

$$\xi \frac{\partial F}{\partial \eta} + \Theta \frac{\partial F}{\partial f} + \pi \frac{\partial F}{\partial p} + k \frac{\partial F}{\partial q} + p \frac{\partial F}{\partial r} = 0 \qquad (4.8)$$

Replacing F by the left side of (4.3) gives:

$$q\Theta - 2\beta p\pi + f\,k + \rho = 0 \qquad (4.9)$$

The functions Θ, π, k, and ρ, given by Eqs. (4.5), are substituted into Eq. (4.9) and we get

$$A_0 + A_1\, q + A_2\, q^2 + A_3\, q^3 = 0 \qquad (4.10)$$

where the variable r in the function ρ, Eq. (4.5), was eliminated by using the differential equation, (4.3), and the A's are given by

$$A_0 = 2\beta pX\overline{W} - fX^2\overline{W} - X^3\overline{W} + \beta(1 - p^2)\ (3X \frac{\partial}{\partial p} + \frac{\partial}{\partial f})\overline{W} \qquad (4.11a)$$

$$A_1 = p \frac{\partial \overline{W}}{\partial p} - \overline{W} - 3X^2 \frac{\partial \overline{W}}{\partial p} - 3X \frac{\partial \overline{W}}{\partial f} + f\ X \frac{\partial \overline{W}}{\partial p} + 3\beta(1-p^2) \frac{\partial^2 \overline{W}}{\partial p^2} \qquad (4.11b)$$

$$A_2 = 2f \frac{\partial^2 \overline{W}}{\partial p^2} - 3X \frac{\partial^2 \overline{W}}{\partial p^2} - 3 \frac{\partial^2 W}{\partial f \partial p} \qquad (4.11c)$$

$$A_3 = \frac{\partial^3 \overline{W}}{\partial p^3} \qquad (4.11d)$$

Since the characteristic function, $\overline{W}$, is independent of q, Eq. (4.10) is satisfied if the coefficients are all equal to zero identically. Thus

$$A_0 = A_1 = A_2 = A_3 = 0 \qquad (4.12)$$

The equation $A_3 = 0$ gives

$$\frac{\partial^3 \overline{W}}{\partial p^3} = 0 \qquad (4.13)$$

which means $\overline{W}$ is quadratic with p, i.e.,

$$\overline{W}(\eta, f, p) = \overline{W}(\eta, f)p^2 + \overline{W}_2 (\eta, f)p + \overline{W}_3(\eta, f) \tag{4.14}$$

This form of $\overline{W}$ can now be substituted into the equation $A_2 = 0$, Eq. (4.12), and the result is

$$(4f\overline{W}_1 - 6 \frac{\partial \overline{W}_1}{\partial \eta} - 3 \frac{\partial \overline{W}_2}{\partial f}) - 12 p \frac{\partial \overline{W}_1}{\partial f} = 0 \tag{4.15}$$

Since both $\overline{W}_1$ and $\overline{W}_2$ are independent of p, Eq. (4.15) leads to

$$p^0: \quad \frac{\partial \overline{W}_1}{\partial f} = 0 \tag{4.16}$$

$$p^1: \quad 4f \overline{W}_1 - 6 \frac{\partial \overline{W}_1}{\partial \eta} - 3 \frac{\partial \overline{W}_2}{f} = 0 \tag{4.17}$$

Eq. (4.16) shows that $\overline{W}_1$ is independent of f, i.e., $\overline{W}_1 = \overline{W}_1(\eta)$. Thus, $\overline{W}_2$ can be found from Eq. (4.17) as

$$\overline{W}_2(\eta, f) = \frac{1}{3} \{2f^2\overline{W}_1 - 6 \overline{W}_1'f + C_1(\eta)\} \tag{4.18}$$

The characteristic function, $\overline{W}$, now takes the form

$$\overline{W}(\eta, f, p) = \overline{W}_1(\eta)p^2 + \frac{1}{3}[2f^2\overline{W}_1(\eta) - 6\overline{W}_1'(\eta)f + C_1(\eta)]p + \overline{W}_3(\eta, f) \tag{4.19}$$

This new form of $\overline{W}$ is now substituted into the condition $A_1 = 0$ which than gives

$$B_0 + B_1 p + B_2 p^2 = 0 \tag{4.20}$$

where

$$B_0 = -\overline{W}_3 - 4f^2 \overline{W}_1'' + 6\overline{W}_1'''f - C_1'' - 3 \frac{\partial^2 \overline{W}_3}{\partial \eta \partial f} + \frac{2}{3} f^3 \overline{W}_1' + \frac{f}{3} C_1' - 6\beta \overline{W}_1 \tag{4.21a}$$

$$B_1 = (6 - 8f)\overline{W}_1' + 6\overline{W}_1'' - 3 \frac{\partial^2 \overline{W}_3}{\partial f^2} \tag{4.21b}$$

$$B_2 = -3\overline{W}_1 (1 + 2\beta) \tag{4.21c}$$

Since both $\overline{W}_1$ and $\overline{W}_3$ are independent of p, Eq. (4.21) gives

$$B_0 = B_1 = B_2 = 0 \tag{4.22a}$$

or,

$$-\overline{W}_3 - 4f^2\overline{W}_1'' + 6\overline{W}_1'''f - C_1'' - 3\frac{\partial^2\overline{W}_3}{\partial\eta\partial f} + \frac{2}{3}f^3\overline{W}_1' + \frac{f}{3}C_1' - 6\beta\overline{W}_1 = 0 \tag{4.22b}$$

$$(6-8f)\overline{W}_1' + 6\overline{W}_1'' - 3\frac{\partial^2\overline{W}_3}{\partial f^2} = 0 \tag{4.22c}$$

$$3\overline{W}_1(1 + 2\beta) = 0 \tag{4.22d}$$

Thus, from Eq. (4.22d),

$$\overline{W}_1 = 0 \tag{4.23}$$

From Eq. (4.22c),

$$\frac{\partial^2\overline{W}_3}{\partial f^2} = 0 \tag{4.24}$$

which gives

$$\overline{W}_3(\eta, f) = \overline{W}_{31}(\eta) + \overline{W}_{32}(\eta)f \tag{4.25}$$

Eq. (4.22b), then becomes

$$-\overline{W}_{31} - \overline{W}_{32}f - C_1'' - 3\overline{W}_{32}' + \frac{f}{3}C_1' = 0 \tag{4.26}$$

which gives

$$-\overline{W}_{31} - C_1'' - 3\overline{W}_{32}' = 0 \tag{4.27a}$$

$$-\overline{W}_{32} + \frac{C_1'}{3} = 0 \qquad (4.27b)$$

since $\overline{W}_{31}$, $\overline{W}_{32}$ and C_1 are functions of η alone.

Eqs. (4.27a) and (4.27b) give

$$\overline{W}_{31} = -2C_1'' \qquad (4.28a)$$

$$\overline{W}_{32} = \frac{1}{3} C_1' \qquad (4.28b)$$

Thus, the characteristic function, $\overline{W}$, becomes

$$\overline{W}(\eta, f, p) = \frac{1}{3} C_1(\eta)p + \frac{1}{3} C_1'f - 2C_1'' \qquad (4.29)$$

Finally, the characteristic function, $\overline{W}$, is substituted into the last condition in Eq. (4.12), namely $A_0 = 0$, which leads to the following equation:

$$D_0 + D_1 f + D_2 p + D_3 f^2 + D_4 f p = 0 \qquad (4.30)$$

where

$$\begin{aligned} D_0 &= 2C_1^{(v)} + \frac{4}{3} C_1'\beta \\ D_1 &= \frac{5}{3} C_1^{(iv)} \\ D_2 &= -4\beta C_1''' - \frac{4}{3} C_1''' \\ D_3 &= -\frac{1}{3} C_1''' \\ D_4 &= (\frac{2}{3}\beta - 1)C_1'' \end{aligned} \qquad (4.31)$$

Since C_1 is a function of η only, Eq. (4.30) gives

$$D_0 = D_1 = D_2 = D_3 = D_4 = 0 \qquad (4.32)$$

If $\beta \neq 0$, then Eq. (4.31) shows that C_1 must be a constant which then leads to the result that

$$\overline{W}(\eta, f, p) = \frac{1}{3} C_1 p \tag{4.33}$$

For the case in which $\beta = 0$, C_1'' is zero, i.e.,

$$C_1(\eta) = C_{11}\eta + C_{12} \tag{4.34}$$

The characteristic function becomes

$$\overline{W}(\eta, f, p) = \frac{1}{3}(C_{11}\eta + C_{12})p + \frac{1}{3} C_{11} f \tag{4.35}$$

As a last step, the finite form of the infinitesimal transformation must be sought. This can be done by using the equation

$$\phi(\eta_1, f_1) = \phi(\eta, f) + \frac{\delta\epsilon}{1!} U\phi + \frac{(\delta\epsilon)^2}{2!} U^2\phi + \text{-------} \tag{4.36}$$

where

$$U\phi = \xi\frac{\partial\phi}{\partial\eta} + \Theta \frac{\partial\phi}{\partial f}$$

Consider first the case in which $\beta \neq 0$ with the characteristic function given in Eq. (4.33). For this case, using Eq. (4.5), the operator U is

$$U = \frac{1}{3} C_1 \frac{\partial}{\partial\eta}$$

By taking ϕ to be η and f respectively, we get:

$$\eta_1 = \eta + \frac{1}{3} C_1 \, \delta\epsilon$$

$$f_1 = f \tag{4.37}$$

Although this group exists under which Eq. (4.1) is invariant, we will not be able to transform the boundary conditions. Thus, the problem cannot be transformed to an initial value problem, unless $\beta = 0$.

For the special case in which $\beta = 0$, the characteristic function is given by Eq. (4.35) which gives the operator U as

$$U = \frac{1}{3}(C_{11}\eta + C_{12}) \frac{\partial}{\partial \eta} - \frac{1}{3} C_{11} f \frac{\partial}{\partial f} \tag{4.38}$$

Again, by putting into Eq. (4.36) $\phi = \eta$ and $\phi = f$, respectively, we get

$$\eta_1 = \eta + \frac{\delta\epsilon}{1!} \frac{1}{3} C_{11}(\eta + \frac{C_{12}}{C_{11}}) + \frac{\delta\epsilon^2}{2!} (\frac{1}{3}C_{11})^2(\eta + \frac{C_{12}}{C_{11}}) + \text{-----}$$

or,

$$\begin{aligned}(\eta_1 + \frac{C_{12}}{C_{11}}) &= (\eta + \frac{C_{12}}{C_{11}})[1 + \frac{\delta\epsilon}{1!} \frac{1}{3} C_{11} + \frac{\delta\epsilon^2}{2!} (\frac{1}{3} C_{11})^2 + \text{-------}] \\ &= (\eta + \frac{C_{12}}{C_{11}})\, e^{\frac{1}{3}C_{11}\delta\epsilon} \end{aligned}\tag{4.39}$$

and

$$\begin{aligned} f_1 &= f + \frac{\delta\epsilon}{1!} (- \frac{1}{3} C_{11})f + \frac{\delta\epsilon^2}{2!} (- \frac{1}{3} C_{11})^2 f + \text{------} \\ &= f[1 - \frac{\delta\epsilon}{1!} \frac{C_{11}}{3} + \frac{\delta\epsilon^2}{2!} (\frac{C_{11}}{3})^2 + \text{------}] \\ &= f\, e^{-\frac{C_{11}}{3} \delta\epsilon} \end{aligned}\tag{4.40}$$

If we put $A = e^{\delta\epsilon}$, Eqs. (4.39) and (4.40) give

$$(\eta_1 + \frac{C_{12}}{C_{11}}) = (\eta + \frac{C_{12}}{C_{11}})\, A^{\frac{1}{3}C_{11}} \tag{4.41a}$$

$$f_1 = f\, A^{-\frac{C_{11}}{3}} \tag{4.41b}$$

This is seen to be the linear group of transformation. For the present case in which the initial condition is given at zero, $C_{12} = 0$.

To finish the analysis, we can follow exactly the same steps as given between Eqs. (2.7) and (2.14).

4.2 APPLICATION TO THE HEAT CONDUCTION EQUATION WITH NONLINEAR HEAT GENERATION

We now consider the heat conduction equation with nonlinear heat generation as follows:

$$\frac{d^2T}{dx^2} + \frac{k+1}{X}\frac{dT}{dx} + f(T) = 0 \tag{4.42}$$

The boundary conditions are

$$\frac{dT(0)}{dx} = 0 \quad \text{and} \quad T(1) = T_0$$

The value of k can be -1, 0 or 1 which correpsonds to plate, cylinder or sphere cases, respectively.

If the following notations are defined:

$$p = \frac{dT}{dx}, \quad q = \frac{d^2T}{dx^2} \tag{4.43}$$

Equation (4.42) then becomes

$$q + \frac{k+1}{x} p + f(T) = 0 \tag{4.44}$$

with the boundary conditions

$$p(0) = 0 \ , \quad T(1) = T_0$$

It is the purpose of this example to find the function f(T) which enables the problem to be transformed to an initial value problem.

Again, an infitesimal transformation is defined as

$$x' = x + (\delta\epsilon)\ \xi(x,T,p)$$

$$T' = T + (\delta\epsilon)\ \theta(x,T,p)$$

$$p' = p + (\delta\epsilon)\ \pi(x,T,p)$$

$$q' = q + (\delta\epsilon)\ \phi(x,T,p,q) \tag{4.45}$$

where, in terms of the characteristic function, $\overline{W}$,

$$\begin{aligned}
\xi &= \frac{\partial\overline{W}}{\partial p} \\
\Theta &= p\,\frac{\partial\overline{W}}{\partial p} - \overline{W} \\
-\pi &= X\overline{W} \\
-\phi &= \left(X^2 + 2qX\,\frac{\partial}{\partial p} + q^2\,\frac{\partial^2}{\partial p^2} + q\,\frac{\partial}{\partial T}\right)\overline{W}
\end{aligned} \tag{4.46}$$

The operator X in Eq. (4.46) is defined as:

$$X = \frac{\partial}{\partial x} + p\,\frac{\partial}{\partial T} \tag{4.47}$$

Next, the condition that the differential equation under investigation be independent of the parameter of transformation (i.e., invariant under the transformation) is introduced. Under the infinitesimal transformation defined in Eq. (4.45),

$$\frac{\partial F}{\partial \epsilon} = 0 \tag{4.48}$$

where F represents the left side of the differential equation defined by Eq. (4.44). Or, in expanded form,

$$\xi\,\frac{\partial F}{\partial x} + \Theta\,\frac{\partial F}{\partial T} + \pi\,\frac{\partial F}{\partial p} + \phi\,\frac{\partial F}{\partial q} = 0 \tag{4.49}$$

Substituting the differential equation, Eq. (4.44), into condition (4.49), we get

$$(q+f)\xi + xf'\Theta + (k+1)\pi + x\phi = 0 \tag{4.50}$$

The functions ξ, Θ, π, and ϕ are now substituted into Eq. (4.50) and we get

$$\{(f+xf'p)\frac{\partial\overline{W}}{\partial p} - xf'\overline{W} - (k+1)\left(\frac{\partial\overline{W}}{\partial x} + p\frac{\partial\overline{W}}{\partial T}\right)$$

$$- x\left(\frac{\partial^2\overline{W}}{\partial x^2} + 2p\frac{\partial^2\overline{W}}{\partial x\partial T} + p^2\frac{\partial^2 W}{\partial T^2}\right)\}$$

$$+ q\{\frac{\partial\overline{W}}{\partial p} - 2x\frac{\partial^2\overline{W}}{\partial x\partial p} - 2xp\frac{\partial^2\overline{W}}{\partial T\partial p} - x\frac{\partial\overline{W}}{\partial T}\} - q^2\{x\frac{\partial^2\overline{W}}{\partial p^2}\} = 0 \quad (4.51)$$

The variable q can now be eliminated by using Eq. (4.44) which gives

$$\{-xf'\overline{W} - (k+1)\frac{\partial\overline{W}}{\partial x} - x\frac{\partial^2\overline{W}}{\partial x^2} + 2xf\frac{\partial^2\overline{W}}{\partial x\partial p} + xf\frac{\partial\overline{W}}{\partial T} - xf^2\frac{\partial^2\overline{W}}{\partial p^2}\}$$

$$+ p\{(xf' - \frac{k+1}{x})\frac{\partial\overline{W}}{\partial p} - 2x\frac{\partial^2\overline{W}}{\partial x\partial T} + 2(k+1)\frac{\partial^2\overline{W}}{\partial x\partial p} + 2xf\frac{\partial^2\overline{W}}{\partial T\partial p} - 2f(k+1)\frac{\partial^2\overline{W}}{\partial p^2}\}$$

$$+ p^2\left\{-x\frac{\partial^2\overline{W}}{\partial T^2} + 2(k+1)\frac{\partial^2\overline{W}}{\partial T\partial p} - \frac{(k+1)^2}{x}\frac{\partial^2\overline{W}}{\partial p^2}\right\} = 0 \quad (4.52)$$

For point transformation* under consideration, the characteristic function, $\overline{W}$, is linear in p, i.e.,

$$\overline{W}(x,T,p) = \overline{W}_1(x,T) + p\,\overline{W}_2(x,T) \quad (4.53)$$

Substituting $\overline{W}$ from Eq. (4.53) into Eq. (4.52), we get

$$a_0 + a_1 p + a_2 p^2 + a_3 p^3 = 0 \quad (4.54)$$

where

$$a_0 = -xf'\overline{W}_1 - (k+1)\frac{\partial\overline{W}_1}{\partial x} - x\frac{\partial^2\overline{W}_1}{\partial x^2} + 2xf\frac{\partial\overline{W}_2}{\partial x} + xf\frac{\partial\overline{W}_1}{\partial T}$$

$$a_1 = (k+1)\frac{\partial\overline{W}_2}{\partial x} - x\frac{\partial^2\overline{W}_2}{\partial x^2} + 3xf\frac{\partial\overline{W}_2}{\partial T} - 2x\frac{\partial^2\overline{W}_1}{\partial x\partial T} - \frac{k+1}{x}W_2$$

$$a_2 = -2x\frac{\partial^2\overline{W}_2}{\partial x\partial T} - x\frac{\partial^2\overline{W}_1}{\partial T^2} + 2(k+1)\frac{\partial\overline{W}_2}{\partial T}$$

$$a_3 = -x\frac{\partial^2\overline{W}_2}{\partial T^2} \quad (4.55)$$

*A point transformation is one whose transformation functions, ξ and Θ, in Eq. (4.45) is independent of p.

Since the a's are functions of x and T only, it follows that

$$a_0 = a_1 = a_2 = a_3 = 0 \qquad (4.56)$$

The condition $a_3 = 0$ gives

$$\frac{\partial^2 \overline{W}_2}{\partial T^2} = 0 \qquad (4.57)$$

which implies that $\overline{W}_2$ is linear in T, i.e.,

$$\overline{W}_2(x,T) = \overline{W}_{21}(x) + \overline{W}_{22}(x)T \qquad (4.58)$$

Substituting this form of $\overline{W}_2$ into the condition $a_2 = 0$, we get

$$-2x\overline{W}_{22}' - x\frac{\partial^2 \overline{W}_1}{\partial T^2} + 2(k+1)\overline{W}_{22} = 0 \qquad (4.59)$$

Integration of Eq. (4.59) with respect to T yields

$$\overline{W}_1 = (-\overline{W}_{22}' + \frac{k+1}{x}\overline{W}_{22})T^2 + \overline{W}_{11}T + \overline{W}_{12} \qquad (4.60)$$

With $\overline{W}_1$ and $\overline{W}_2$ given by Eqs. (4.60) and (4.58), the condition $a_1 = 0$ leads to the following equation:

$$3xf\overline{W}_{22} + \{(k+1)\overline{W}_{21}' - x\,\overline{W}_{21}'' - 2x\overline{W}_{11}' - \frac{k+1}{x}\overline{W}_{21}\}$$

$$+ 3T\{xW_{22}'' - (k+1)W_{22}' + \frac{(k+1)}{x}W_{22}\} = 0 \qquad (4.61)$$

Equation (4.61) shows that f must be a linear function of T which is impossible under the assumption made at the outset that f is a nonlinear function. The only possibility remaining is

$$\overline{W}_{22} = 0 \qquad (4.62)$$

With this, Eq. (4.61) is reduced to:

$$(k+1)\overline{W}_{21}' - x\,\overline{W}_{21}'' - \frac{k+1}{x}\,\overline{W}_{21} - 2x\overline{W}_{11}' = 0 \tag{4.63}$$

Finally, the condition $a_0 = 0$ gives

$$f' - \frac{b_0}{\overline{W}_{11}T + \overline{W}_{12}}\, f = -\frac{b_1 T + b_2}{\overline{W}_{11}T + \overline{W}_{12}} \tag{4.64}$$

where

$$b_0 = 2\overline{W}_{21}' + \overline{W}_{11} \tag{4.65}$$

$$b_1 = -(\frac{k+1}{x}\,\overline{W}_{11}' + \overline{W}_{11}'') \tag{4.66}$$

$$b_2 = -(\frac{k+1}{x}\,\overline{W}_{12}' + \overline{W}_{12}'') \tag{4.67}$$

Equation (4.63) and (4.64) can now be used to determine the functional forms of $f(T)$ for which Eq. (4.42) can be reduced to an initial value problem. We now consider the following case:

Case 1 $\overline{W}_{11} \neq 0$, $b_0 \neq 0$

Using an integrating factor

$$e^{\int p\,dx} = (\overline{W}_{11}T + \overline{W}_{12})^{-b_0/\overline{W}_{11}}, \tag{4.68}$$

the solution to Eq. (4.64) can be written as

$$f = (\overline{W}_{11}T+\overline{W}_{12})^{b_0/\overline{W}_{11}} \left\{ -\int \frac{b_1}{\overline{W}_{11}} (\overline{W}_{11}T+\overline{W}_{12})^{-b_0/\overline{W}_{11}}\, dT \right.$$
$$\left. - \int (b_2 - b_1\frac{\overline{W}_{12}}{\overline{W}_{11}})(\overline{W}_{11}T+\overline{W}_{12})^{-(b_0/\overline{W}_{11}+1)}\, dT + C_1 \right\} \tag{4.69}$$

For the case in which $b_0/\overline{W}_{11}$ does not equal to unity, Eq. (4.69) gives

$$f = \frac{b_2}{b_0} - \frac{b_1(\overline{W}_{12}+b_0T)}{b_0(\overline{W}_{11}-b_0)} + C_1(\overline{W}_{11}T+\overline{W}_{12})^{b_0/\overline{W}_{11}} \tag{4.70}$$

Recalling that the heat generation function, f, is defined as a function of T only, the Coefficients must be independent of x, i.e., constants. Thus, we put

$$\frac{b_2}{b_0} = C_2 \,, \quad \frac{b_1 \ \overline{W}_{12}}{b_0(W_{11} - b_0)} = C_3 \,, \quad \overline{W}_{11} = C_4 \,,$$

$$\overline{W}_{12} = C_5 \,, \quad \frac{b_0}{\overline{W}_{11}} = C_6 \tag{4.71}$$

The conditions $\overline{W}_{11} = C_4$ and $\overline{W}_{12} = C_5$ indicate that $b_1 = b_2 = 0$, based on Eqs. (4.66) and (4.67). This in turn means $C_2 = C_3 = 0$. The last condition gives $b_0 = C_6 C_4$. From Eq. (4.65), we get

$$C_6 C_4 = 2\overline{W}'_{21} + C_4$$

Thus,

$$\overline{W}_{21} = \frac{C_4(C_6-1)}{2} x + C_7 \tag{4.72}$$

Substituting W_{21} from Eq. (4.72) into the condition given by Eq. (4.63), we get $C_7 = 0$. As a summary, the following functions are obtained:

$$b_0 = C_6 C_4 \,, \quad b_1 = b_2 = 0 \,, \quad \overline{W}_{11} = C_4$$

$$\overline{W}_{12} = C_5 \,, \quad \overline{W}_{21} = \frac{C_4(C_6-1)}{2} x \,, \ \overline{W}_{22} = 0 \tag{4.73}$$

Thus, the heat generation function f(T), takes the form

$$f(T) = C_1(C_4 T + C_5)^{C_6} \tag{4.74}$$

and the characteristic function becomes

$$\overline{W}(x,T,p) = (C_4 T + C_5) + p \left[\frac{C_4(C_6-1)}{2} x\right] \tag{4.75}$$

The transformation functions, ξ and Θ, in Eq. (4.45) become

$$\xi = \frac{\partial W}{\partial p} = \frac{C_4(C_6-1)}{2} x \tag{4.76}$$

$$\Theta = p\frac{\partial W}{\partial p} - W = -(C_4T + C_5) \tag{4.77}$$

The infinitesimal transformation, Eq. (4.45), can thus be represented by (see Reference 6)

$$Uf = \frac{C_4(C_6-1)}{2}x \frac{\partial f}{\partial x} - (C_4T + C_5)\frac{\partial f}{\partial T} \tag{4.78}$$

As a final step, the finite form of the infinitesimal transformation, (4.45), is sought. This can be done by using the equation (cf. Reference 6, p. 44-46):

$$f(x_1, y_1) = f(x,y) + \frac{\delta\epsilon}{1!} Uf + \frac{(\delta\epsilon)^2}{2!} U^2 f + \text{------} \tag{4.79}$$

By putting f = x, Eq. (4.79) becomes

$$x_1 = x + \frac{\delta\epsilon}{1!}\frac{C_4(C_6-1)}{2}x + \frac{(\delta\epsilon)^2}{2!}\left[\frac{C_4(C_6-1)}{2}\right]^2 x + \text{-----}$$

$$= x\left[1 + \frac{1}{1!}\frac{C_4(C_6-1)\delta\epsilon}{2} + \frac{1}{2!}\frac{C_4{}^2(C_6-1)^2(\delta\epsilon)^2}{2} + \text{-----}\right]$$

or,

$$x_1 = x \exp\left[\frac{C_4(C_6-1)\delta\epsilon}{2}\right] = x\, A^{C_4(C_6-1)/2} \tag{4.80}$$

where $A = \exp(\delta\epsilon)$. By putting f = T, Eq. (4.79) gives

$$T_1 = T + \frac{\delta\epsilon}{1!}[-(C_4T+C_5)] + \frac{(\delta\epsilon)^2}{2!}[-(C_4T+C_5)](-C_4) + \text{-----}$$

$$= T + \frac{\delta\epsilon[-C_4]}{1!}\left(T + \frac{C_5}{C_4}\right) + \frac{[\delta\epsilon(-C_4)]^2}{2!}\left(T + \frac{C_5}{C_4}\right) + \text{-----}$$

or,

$$T_1 + \frac{C_5}{C_4} = (T + \frac{C_5}{C_4}) \left\{ 1 - \frac{(\delta\epsilon)(C_4)}{1!} + \frac{(\delta\epsilon)^2(C_4)^2}{2!} ----- \right\}$$

or,

$$T_1 + \frac{C_5}{C_4} = (T + \frac{C_5}{C_4}) \exp\{-C_4\delta\epsilon\} = (T + \frac{C_5}{C_4}) A^{-C_4} \tag{4.81}$$

Thus, the finite transformation is given by

$$x_1 = x A^{C_4(C_6-1)/2}, \quad T_1 + \frac{C_5}{C_4} = (T + \frac{C_5}{C_4})A^{-C_4} \tag{4.82}$$

This is seen to be linear group of transformation discussed in Section 2.1 for the case of a power law heat generation.

Consider next the case in which the ratio $b_0/\overline{W}_{11} = 1$. The solution to f in Eq. (4.69) than becomes

$$f = -(\overline{W}_{11}T+\overline{W}_{12}) \frac{b_1}{\overline{W}_{11}^2} \ell n(\overline{W}_{11}T+\overline{W}_{12}) + (b_2-b_1 \frac{\overline{W}_{12}}{\overline{W}_{11}}) + C_1(\overline{W}_{11}T+\overline{W}_{12}) \tag{4.83}$$

Again, since the heat generation function, f, is only a function of the temperature T, the coefficients must all be constants, i.e.,

$$\overline{W}_{11} = C_2, \quad b_1 = C_3, \quad \overline{W}_{12} = C_4, \quad b_2 = C_5, \tag{4.84}$$

The conditions $\overline{W}_{11} = C_2$ and $\overline{W}_{12} = C_4$ indicate that $b_1 = b_2 = 0$, based on Eqs. (4.66) and (4.67), which in turn means $C_3 = C_5 = 0$. Eq. (4.65) now gives

$$\overline{W}_{21}' = 0$$

so, we can write $\overline{W}_{21} = C_6$. Substituting $\overline{W}_{21} = C_6$ into the last condition, Eq. (4.63), we conclude that $C_6 = 0$. The heat generation function is thus:

$$f = C_1(C_2T+C_4) \tag{4.85}$$

and the characteristic function become

$$\overline{W} = C_2 T + C_4 \qquad (4.86)$$

By following the same steps as before, the finite transformation is found to be

$$x_1 = x\,, \quad \left(T_1 + \frac{C_4}{C_2}\right) = \left(T + \frac{C_4}{C_2}\right) A^{C_2} \qquad (4.87)$$

This case is a special case of "power-law heat generation" discussed above for a power of unity. It is to be noted that the same form of $\overline{W}$ can be obtained by putting C_6 equal to 1 in Eq. (4.82).

Case 2 $b_0 = 0$

For this case, integration of Eq. (4.69) gives

$$f = -\frac{b_1}{\overline{W}_{11}} T - \frac{b_2}{\overline{W}_{11}} \ln(\overline{W}_{11} T + \overline{W}_{12}) + C_1 \qquad (4.88)$$

Again, the coefficients in (4.88) should be constants, i.e.,

$$b_1 = C_2\,, \quad b_2 = C_3\,, \quad \overline{W}_{11} = C_4\,, \quad \overline{W}_{12} = C_5$$

The conditions $\overline{W}_{11} = C_4$ and $\overline{W}_{12} = C_5$ again lead to $C_2 = C_3 = 0$ (or $b_1 = b_2 = 0$). Equation (4.65) gives

$$2\overline{W}_{21}' + C_4 = 0$$

and thus

$$\overline{W}_{21} = -\frac{C_4}{2} x + C_6$$

The condition, (4.63), then gives $C_6 = 0$. The heat generation function and the characteristic function are therefore as follows:

$$f = C_1 \qquad (4.89)$$

$$\overline{W} = C_4T + C_5 + p[-\frac{C_4}{2}x] \tag{4.90}$$

This is seen to be a special case of case 1, and can be obtained by putting $C_6 = 1$ into Eqs. (4.74) and (4.75).

Case 3 $W_{11} = 0$

The differential equation for f becomes

$$f' - C_2 f = C_3 \tag{4.91}$$

in conformity with the requirement that f be independent of x. The constants, C_2 and C_3, are

$$\frac{b_0}{\overline{W}_{12}} = C_2 \quad \text{and} \quad \frac{b_2}{\overline{W}_{12}} = C_3 \tag{4.92}$$

From Eqs. (4.65) and (4.67), Eq. (4.92) can be written as

$$C_2\overline{W}_{12} = 2\overline{W}_{21}' \tag{4.93}$$

and

$$C_3\overline{W}_{12} = -\frac{k+1}{x}\overline{W}_{12}' - \overline{W}_{12}'' \tag{4.94}$$

Also, from condition (4.63), another relation is obtained as:

$$(k+1)\overline{W}_{21}' - x\overline{W}_{21}'' - \frac{k+1}{x}\overline{W}_{21} = 0 \tag{4.95}$$

Eq. (4.95) now gives

$$\overline{W}_{21} = C_4x^{k+1} + C_5x \tag{4.96}$$

Substitution of $\overline{W}_{21}$ into Eq. (4.93) yields, after integration, the solution of $\overline{W}_{12}$:

$$\overline{W}_{12} = \frac{2(k+1)C_4}{C_2} x^k + \frac{2C_5}{C_2} \qquad (4.97)$$

Finally, $\overline{W}_{12}$ from Eq. (4.97) is substituted into Eq. (4.94) and we get

$$C_3 C_4(k+1)X^k + 2k(k+1)C_4 X^{k-2} + C_3 C_5 = 0 \qquad (4.98)$$

To satisfy this condition, six possibilities exist, namely,

1. $k = 0$
   1a: $C_3 = 0$
   1b: $C_4(k+1) + C_5 = 0$
2. $k = -1$
   2a: $C_3 = 0$
   2b: $C_5 = 0$
3. $k \neq 0$, $k \neq -1$
   3a: $C_3 = 0$, $C_4 = 0$
   3b: $C_5 = 0$, $C_4 = 0$

Using the same technique as before, the final form of the heat generation function f, the characteristic function, $\overline{W}$, and the finite transformation are found to be as follows:

1. $k = 0$

   1a: $C_3 = 0$

$$f = C_6 e^{C_2 T} \qquad (4.99)$$

$$\overline{W} = 2\frac{C_4+C_5}{C_2} + p(C_4+C_5)x \qquad (4.100)$$

$$x_1 = xe^{(C_4+C_5)\delta\epsilon} \quad \text{and} \quad T_1 = T - \frac{2}{C_2}(C_4+C_5)\delta\epsilon \qquad (4.101)$$

   1b. $C_4 + C_5 = 0$

$$f = -\frac{C_3}{C_2} + C_6 e^{C_2 T}$$

$W = 0$, which means only the identical transformation will be applicable transformation.

2. $k = -1$

   2a: $C_3 = 0$

$$f = C_6 e^{C_2 T} \qquad (4.102)$$

$$W = 2\frac{C_5}{C_2} + p(C_4 + C_5 x) \qquad (4.103)$$

$$(x_1 + \frac{C_4}{C_5}) = (x + \frac{C_4}{C_5})\, e^{C_5\delta\epsilon} \quad \text{and } T_1 = T - \frac{C_5}{C_2}\delta\epsilon \qquad (4.104)$$

2b: $$f = -\frac{C_3}{C_2} + C_6\, e^{C_2 T} \qquad (4.105)$$

$$W = C_4 p \qquad (4.106)$$

$$x_1 = x + C_4\delta\epsilon\ , \quad T_1 = T \qquad (4.107)$$

3. $k \neq 0\ ,\ k \neq -1$

$C_3 = 0, \quad C_4 = 0$

$$f = C_6\, e^{C_2 T} \qquad (4.108)$$

$$W = 2\frac{C_5}{C_2} + p(C_5 x) \qquad (4.109)$$

$$x_1 = x e^{C_5\delta\epsilon}\ , \quad T_1 = T - 2\frac{C_5}{C_2}\delta\epsilon \qquad (4.110)$$

3b. $C_5 = 0, \quad C_4 = 0$

Again, the result is the identical transformation, as in case 1b.

Details from this point on and the limitations to the method are the same as in Section 2. They will not be repeated here.

4.3 CONCLUDING REMARKS

The method developed in this section is seen to be very general and, like the simple group-theoretic method, only algebraic equations need to be solved. The general method is, however, considerably longer than the simple group-theoretic method. For a given ordinary differential equation, therefore, it is prefered to try the simple group-theoretic method first. If this method fails, the general method is then applied and the possible group of transformations searched. For certain problems, e.g., the example given in Section 4.2, only the general method can provide the answer. In case both methods fail, we can conclude that the problem cannot be transformed to an initial value problem.

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